Exercise 1. (5 pts) You can directly write an answer for (a),(b),(c). Justify your answer for (d).

(a) (1 pt) A class of 10 students wants to form a music band with 3 distinct players and each student is willing to learn any instrument. How many different compositions of a band with one guitar, one bass, and one drum can they form?

(b) (1 pt) Same question as in (a) for a band with two identical guitars, one bass, and one drum.

(c) (1 pt) Let X and Y be i.i.d. uniform random variables on $\{1, 2, \ldots, 100\}$. What is the probability that $X = Y$?

(d) (2 pts) Among 1000 coins, 1 coin has heads on both sides and the other 999 coins are fair (i.e., head or tale with probability 1/2). You pick one coin uniformly at random, and toss it 10 times. Each time, the coin turns up head. What is the probability that the coin you picked is the unfair one?

Exercise 2. (5 pts) For each question, justify statements and counter-examples where appropriate.

(a) (1 pt) Let (X_1, X_2) be a multivariate normal random vector, where each component has mean 0 and variance 1, and $\mathrm{Cov}(X_1, X_2) = \frac{1}{\sqrt{2}}$. Let $W = \sqrt{2}X_1 - X_2$. Are W and X_2 independent?

(b) (2 pts) Let X_1, X_2 be i.i.d. random variables each with mean 0 and variance 1. Let $X_+ = X_1 + X_2$ and $X_- = X_1 - X_2$. Are X_+ and X_- always dependent or always independent? Answer both questions.

(c) (1 pt) Can three random variables form a random vector that is multivariate normal such that any two random variables are independent, but the three random variables together are not mutually independent?

(d) (1 pt) Same question as in (c) if the random vector is not required to be multivariate normal.

Exercise 3. (5 pts) Let $X_1, ..., X_n$ be iid random variables such that $P(X_i = 2) = P(X_i = -2) = 1/2$ for all $i \in \{1, ..., n\}$.

(a) (2 pts) Prove that $P(\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{1}{100})$ tends to 0 as n tends to infinity.

(b) (1 pt) Consider now the random variable $\tilde{S}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$. Compute its moment-generating function, $\mathcal{M}_{\tilde{S}_n}(t) = E(e^{t\tilde{S}_n})$.

(c) (2 pts) Using $\mathcal{M}_{\tilde{S}_n}(t)$, find the limiting distribution of $\tilde{S}_n$, i.e., find the distribution of a random variable Z such that $\tilde{S}_n \xrightarrow{D} Z$. Justify your answer.

Hint: For large n, you can use the approximation $\left(\frac{1+\exp(\frac{x}{\sqrt{n}})}{2}\right)^n \simeq \exp(\frac{x\sqrt{n}}{2} + \frac{x^2}{8})$.

Exercise 4. (5 pts) Let $X_1, ..., X_n \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$, with $\theta > 0$.

(a) (1 pt) Derive the estimator $\widehat{\theta}_1$ of θ using the method of moments.

(b) (2 pts) Derive the estimator $\widehat{\theta}_2$ of θ using the maximum likelihood method.

(c) (2 pts) Compute the bias, the variance, and the mean square error of $\widehat{\theta}_2$. Recall that MSE $= \mathrm{E}((\widehat{\theta}_2 - \theta)^2)$.

Exercise 5. (5 pts) Let 0^n be the vector of length n that contains only 0's, and let 1^n be the vector of length n that contains only 1's. Assume that n is even.

Consider the following hypothesis testing problem. Under H_0, the random vector Y^n is the output of transmitting 0^n on a binary symmetric channel, i.e., each component of 0^n is flipped independently to a 1 with a probability $p \in (0, 1/2)$ to produce the random vector Y^n. Under H_1, the random vector Y^n is the output of transmitting 1^n on the same channel, i.e., each component of 1^n is flipped independently to a 0 with probability p. Upon observing Y^n, we would like to decide between the hypotheses H_0 and H_1.

Let P_e be the least possible value of the sum of the false positive and false negative error probabilities for this hypothesis testing problem. Recall that the false positive refers to declaring H_1 when the true hypothesis is H_0, and the false negative refers to the reverse case.

(a) (2 pts) Give an optimal test, i.e., how to choose between H_0 and H_1 given Y^n in order to achieve P_e. Justify your answer.

(b) (2 pts) Prove that P_e tends to 0 when n tends to infinity.

(c) (1 pt) Does P_e still tend to 0 when n tends to infinity if p depends on n and is given by $p = \frac{1}{2} - \frac{1}{\log(n)}$? Justify your answer.